Semi-analytical Approach for Sky Localization of GWs

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Overview

- Problem & Motivation
- Bayesian Method: Prior setting and Marginalization
- Case Study
- Summary

Localization and Marginalization

- Posterior ∝ Prior * Likelihood

$$\underbrace{p(\boldsymbol{\vartheta} \mid \mathbf{d}(t))}_{\text{Posterior}} = \frac{\overbrace{p(\mathbf{d}(t) \mid \boldsymbol{\vartheta})}^{\text{Likelihood}} \overbrace{p(\boldsymbol{\vartheta})}^{\text{Prior}}}{\underbrace{p(\mathbf{d}(t))}_{\text{Evidence}}},$$

- Localization needs 5-fold integral over extrinsic parameters, like Bayestar *(L. Singer+, arXiv:1508.03634) uses numerical integration

$$\int p(\alpha, \delta, \underline{r, \iota, t_c, \psi, \phi_c}) dr d\iota dt_c d\psi d\phi_c \quad = p_{marginalized}(\alpha, \delta) \text{ -> sky direction}$$

- Can we marginalize those parameters analytically?

*There are also many other works on fast localization, say, Chen+arXiv:1509.00055, Tsutsui+2005.08163, Cornish arXiv:2101.01188...

Bayes I: Preparation - parameter set

Parameter set

- The response of the i-th detector can be re-expressed as

$$h^{(i)} = (G_+^{(i)}, G_\times^{(i)})\mathbf{A_c} h_c + (G_+^{(i)}, G_\times^{(i)})\mathbf{A_s} h_s,$$

$$\mathbf{A} = \begin{pmatrix} \mathbf{A_c} & \mathbf{A_s} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$
$$= \frac{1\mathrm{Mpc}}{r} \begin{pmatrix} \cos 2\psi & \sin 2\psi \\ -\sin 2\psi & \cos 2\psi \end{pmatrix} \begin{pmatrix} \frac{1+\cos^2 \iota}{2} & \\ & \cos \iota \end{pmatrix} \begin{pmatrix} \cos \phi_c & \sin \phi_c \\ -\sin \phi_c & \cos \phi_c. \end{pmatrix}$$

$$\vartheta = \{\alpha, \delta, t_c, \underline{r, \iota, \phi_c, \psi}\}.$$

Substitution

$$\vartheta = \{\alpha, \delta, t_c, \underline{A_{11}, A_{12}, A_{21}, A_{22}}\}.$$

- Same as Bayestar: The errors of the intrinsic parameters are semi-independent from errors in sky localization. To achieve low-latency online localization, we assume intrinsic parameters from online matched filtering search are accurate.

Likelihood

- Likelihood for detector network (Gaussian noise): a function of (α, δ, tc, **A**)

$$p(\mathbf{d} \mid \boldsymbol{\vartheta}) \propto e^{(\mathbf{d}^{\mathbf{T}}|\mathbf{h}) - \frac{1}{2}(\mathbf{h}^{\mathbf{T}}|\mathbf{h})}$$

$$\propto \prod_{x=\{c,s\}} e^{(\mathbf{d}^{\mathbf{T}}|\mathbf{H}_{\mathbf{x}})\mathbf{G}_{\sigma}\mathbf{A}_{\mathbf{x}} - \frac{1}{2}\mathbf{A}_{\mathbf{x}}^{\mathbf{T}}\mathbf{G}_{\sigma}^{\mathbf{T}}\mathbf{G}_{\sigma}\mathbf{A}_{\mathbf{x}}}$$

$(\mathbf{d}^{\mathbf{T}}|\mathbf{H}_{\mathbf{x}})$: SNR timeseries, (…|…) denotes inner product

$\mathbf{G}_{\sigma}$: Weighted response matrix (weighted by horizon distance)

$\mathbf{A}_{\mathbf{x}}$: x={c,s}, **Ac** & **As** denote left & right half of matrix **A**

Prior functions for α, δ, tc

$$\{\alpha, \delta, t_c, A_{11}, A_{12}, A_{21}, A_{22}\}$$

- Prior functions:
  - General prior functions for α, δ, tc

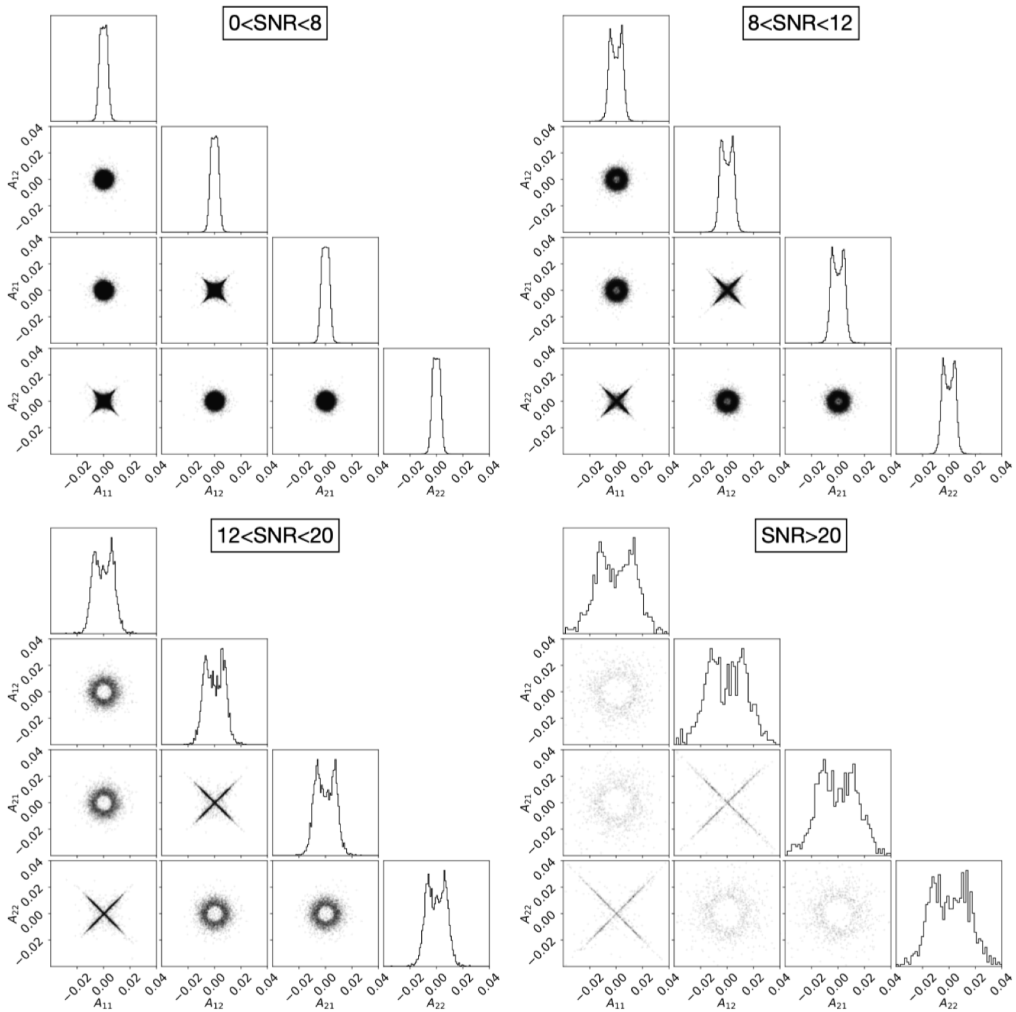
$$\begin{aligned} &p(\alpha) \propto 1, && 0 < \alpha < 2\pi \\ &p(\sin\delta) \propto 1, && -1 < \sin\delta < 1 \\ &p(t_c) \propto 1, && t_{trigger} - T < t_c <, t_{trigger} + T, \\ & && T = 20ms \end{aligned}$$

  - Using injections to investigate **A**' s distribution

$$\mathbf{A} = \begin{pmatrix} \mathbf{A_c} & \mathbf{A_s} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$
$$= \frac{1\mathrm{Mpc}}{r} \begin{pmatrix} \cos 2\psi & \sin 2\psi \\ -\sin 2\psi & \cos 2\psi \end{pmatrix} \begin{pmatrix} \frac{1+\cos^2\iota}{2} & \\ & \cos\iota \end{pmatrix} \begin{pmatrix} \cos\phi_c & \sin\phi_c \\ -\sin\phi_c & \cos\phi_c . \end{pmatrix}$$

Prior distribution of **A**: Simulation & Conclusions

- Simulation of 50000 BNS events in 3 detectors (HLV) with Gaussian noise colored with O2 sensitivity
- Easy to find:
  - Symmetric bimodal distribution
  - Location of peaks varies with SNR*
  - Correlation between (A11,A22), (A12, A21)



*Here SNR means network SNR, i.e., $SNR_{net} = \sqrt{SNR_H^2 + SNR_L^2 + SNR_v^2}$

Prior distribution of **A**: Bimodal distribution

- The following function can be used to approximate bimodal function:

$$p(A_{ij}) \propto e^{-\frac{(A_{ij}-\mu)^2}{2\sigma^2}} + e^{-\frac{(A_{ij}+\mu)^2}{2\sigma^2}}$$

$i, j = 1,2,$ A_{ij} represents elements of **A**

- Why bimodal? Selection effect.
  - The amplitude of GWs increases monotonically with $|\cos\iota|$ and $1/r$

$$h_+(t) = \frac{1}{2}a(t)(1+\cos^2\iota)\cos(\phi(t)+\phi_c)$$
$$h_\times(t) = a(t)\cos\iota\sin(\phi(t)+\phi_c)\,.$$

$$a(t) \propto 1/r$$

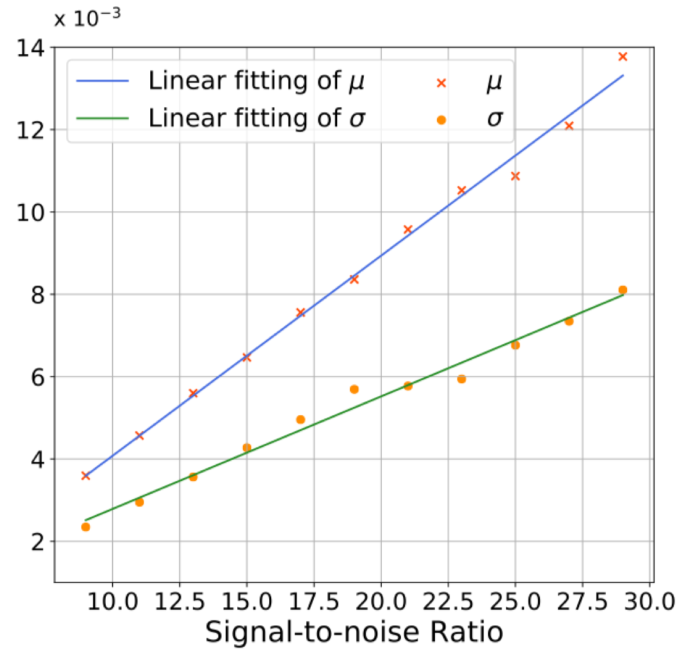
  - When we select GW events in a higher SNR range, sources with larger $|\cos\iota|$ values and smaller r are more likely to be selected.
  - $|\cos\iota|$ tends to be 1, which means a symmetric bimodal distribution of $\cos\iota$, hence Aij are also bimodally distributed、

Prior distribution of **A**: Dependence on SNR

$$p(A_{ij}) \propto e^{-\frac{(A_{ij}-\mu)^2}{2\sigma^2}} + e^{-\frac{(A_{ij}+\mu)^2}{2\sigma^2}}$$

$$\mu = 0.0004860 \text{ SNR} - 0.0007827,$$
$$\sigma = 0.0002733 \text{ SNR} + 0.00005376.$$

- In each SNR range, using least square method to get the best-fit μ and σ
- We can fit μ, σ~SNR relation for each PSD
- Why depends on SNR? Selection effect
  - Larger SNR -> smaller distance r
  - Aij ∝ 1/r
  - When selecting higher SNR events, |Aij| also getting larger, thus peaks in Aij’ s distribution move outside

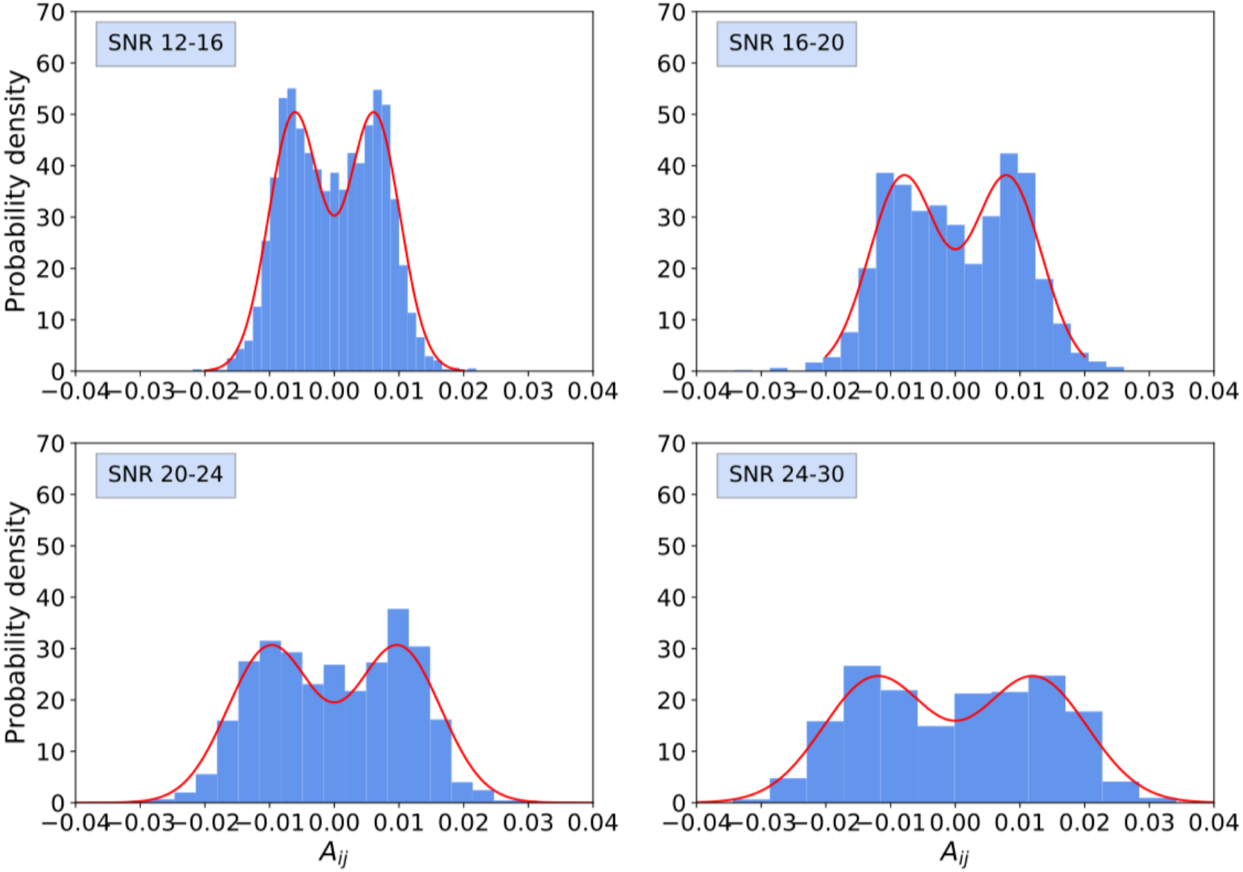


Prior distribution of **A**: Dependence on SNR

An example for O2 sensitivity:

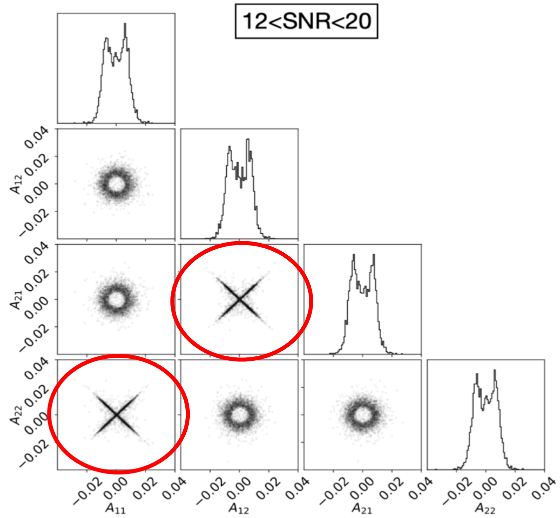
$$p(A_{ij}) \propto e^{-\frac{(A_{ij}-\mu)^2}{2\sigma^2}} + e^{-\frac{(A_{ij}+\mu)^2}{2\sigma^2}}$$

$$\mu = 0.0004860\ \mathrm{SNR} - 0.0007827,$$
$$\sigma = 0.0002733\ \mathrm{SNR} + 0.00005376.$$



Prior distribution of **A**: Correlation on diagonal lines

- In the corner plot, A11&A22, A21&A12 are correlated (see the ○)
- Why correlated?
  - Detectable GW events usually have large |cos ι| (Tsutsui+2005.08163)
  - From Aij' s expression:
    - cos ι ->1, A11->A22, A12->-A21
    - cos ι ->-1, A11->-A22, A12->A21
  - cos ι has half chance to be 1, another half to be -1, thus A11 has equal chance to be +A22 or -A22
  - This causes a "cross" in the corner plot



$$A_{11} = \frac{1}{r}\left(\frac{1+\cos^2\iota}{2}\cos 2\psi\cos\phi_c - \cos\iota\sin 2\psi\sin\phi_c\right),$$

$$A_{21} = -\frac{1}{r}\left(\frac{1+\cos^2\iota}{2}\sin 2\psi\cos\phi_c + \cos\iota\cos 2\psi\sin\phi_c\right),$$

$$A_{12} = \frac{1}{r}\left(\frac{1+\cos^2\iota}{2}\cos 2\psi\sin\phi_c + \cos\iota\sin 2\psi\cos\phi_c\right),$$

$$A_{22} = -\frac{1}{r}\left(\frac{1+\cos^2\iota}{2}\sin 2\psi\sin\phi_c - \cos\iota\sin 2\psi\cos\phi_c\right).$$

Prior distribution of **A**: Correlation on diagonal lines

- Approximation: |cos ι|->1, we have A22 = ±A11, i.e.

$$p(A_{22}|A_{11}) = \frac{\delta(A_{22} - A_{11}) + \delta(A_{22} + A_{11})}{2}$$

$$\delta(0) = 1,$$
$$\delta(x) = 0 \; for \; x \neq 0$$

(the same for A12&A21)

The approximation A22 = ±A11 is also used in Tsutsui+2005.08163

- Following rules of conditional probability:

$$\begin{aligned} p(\mathbf{A}) &= p(A_{11}, A_{12}, A_{21}, A_{22}) \\ &= p(A_{11}, A_{22})p(A_{21}, A_{12}) \\ &= p(A_{11})p(A_{22}|A_{11})p(A_{21})p(A_{12}|A_{21}) \end{aligned}$$

Semi-analytical marginalization

- Gaussian integral:

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi} \quad \Rightarrow \quad \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x}+\mathbf{B}^{\mathbf{T}}\mathbf{x}} d^2\mathbf{x} = \sqrt{\frac{(2\pi)^2}{\det \mathbf{A}}} e^{\frac{1}{2}\mathbf{B}^{\mathbf{T}}\mathbf{A}^{-1}\mathbf{B}}$$

- Marginalization of **A** can be converted to a 16 Gaussian integrals…

Likelihood

$$\begin{aligned} &p(\alpha, \delta \mid \mathbf{d}) \\ &\propto \int_{t_{\mathrm{trigger}}-T}^{t_{\mathrm{trigger}}+T} dt_c \int d^4\mathbf{A} \exp\left\{ \sum_{x=\{c,s\}} -\frac{1}{2}\mathbf{A}_{\mathbf{x}}^{\mathrm{T}}\mathbf{M}\mathbf{A}_{\mathbf{x}} + \mathbf{J}_{\mathbf{x}}^{\mathrm{T}}\mathbf{A}_{\mathbf{x}} \right\} \\ &\times p(A_{11})p(A_{22}|A_{11})p(A_{21})p(A_{12}|A_{21}) \\ &\propto \int_{t_{\mathrm{trigger}}-T}^{t_{\mathrm{trigger}}+T} dt_c (I_1 + I_2 + I_3 + I_4). \end{aligned}$$

Prior probability

1-fold numerical integral

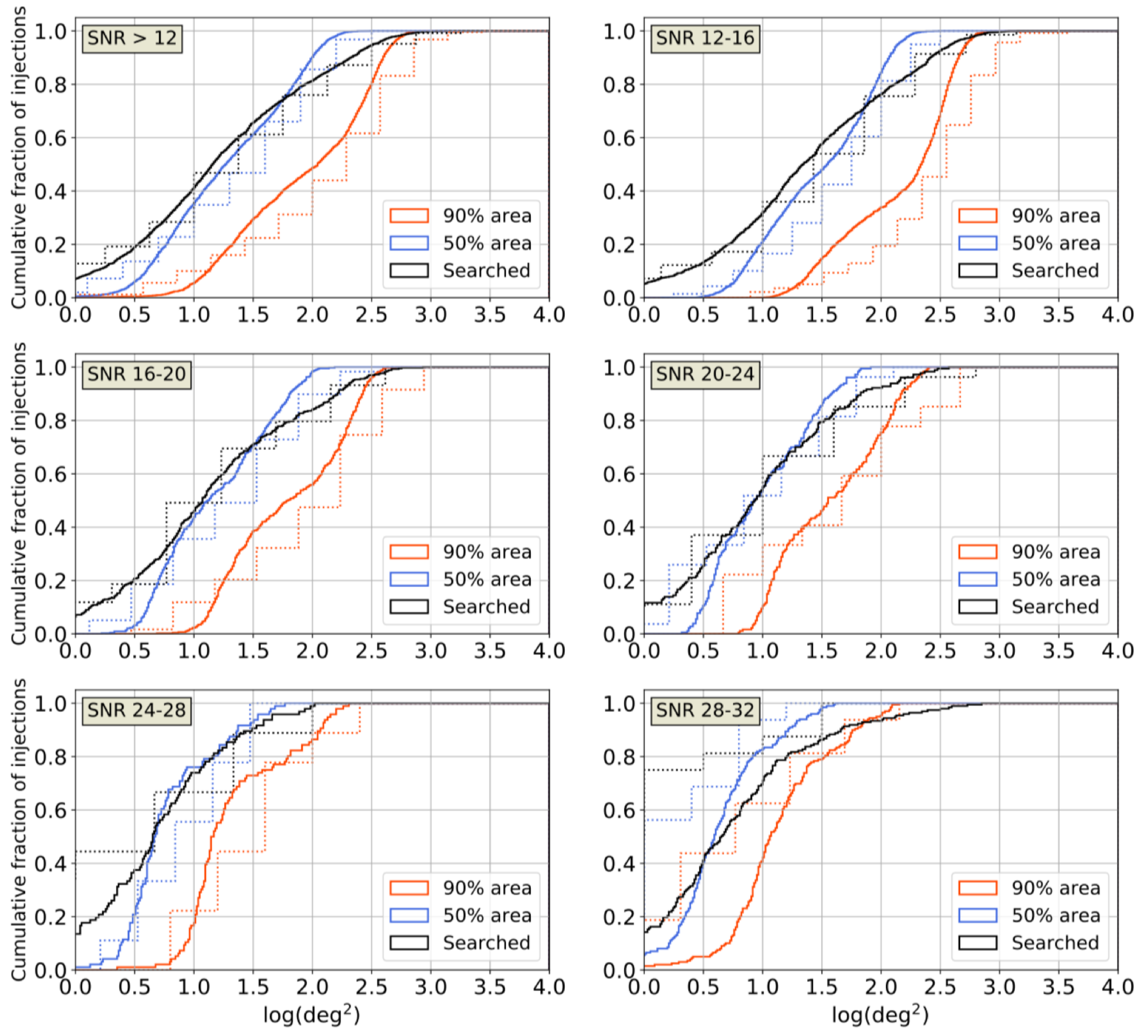
Analytical results of 16 Gaussian integrals
Input: SNR timeseries & horizon distance of each detector

Case Study I: Injection

Cumulative distribution of areas

- Two kinds of areas:
  - 90%/50% confidence area
  - Searched area
- Compare our results with LALInference*
- Performance of the two method are very close

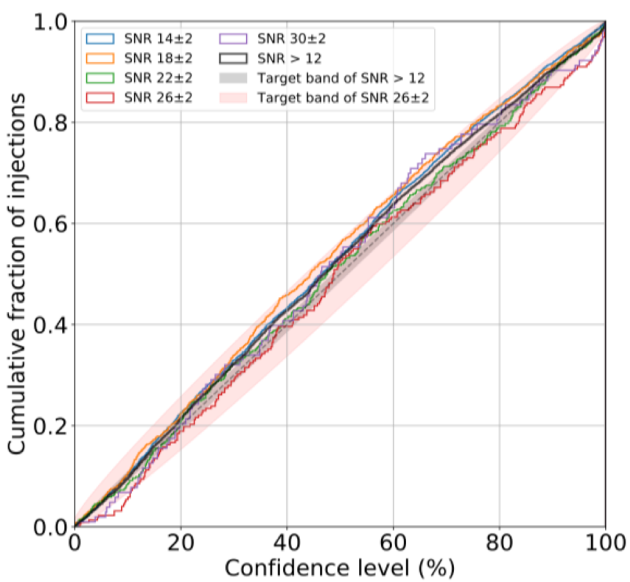
Solid line: Our method;
Dashed line: LALInference



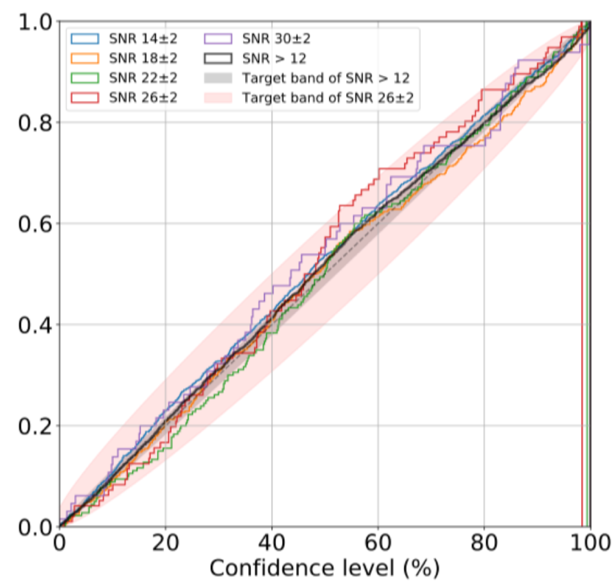
*data for LAL is from https://git.ligo.org/leo-singer/first2years-gstlal-subset-data/

Self-consistency

- When we claim a x% confidence area, we should use this method to correctly answer x% of test events.
- Plot confidence level vs fraction of injections that we answer right, we should get a diagonal line (p-p plot)



Design noise

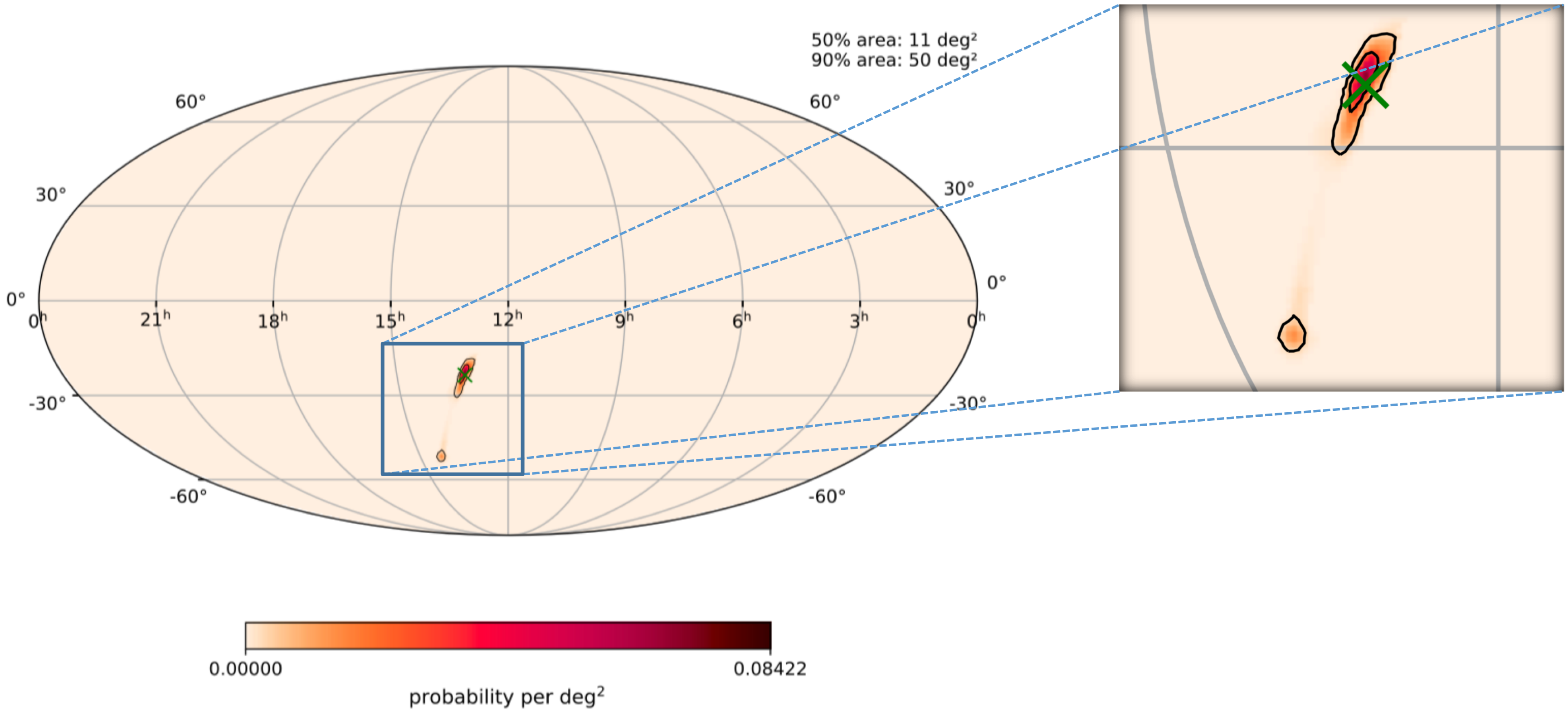


O2 noise

Case Study II: Real event

Test on GW170817

- The detected optical counterpart, (α, δ) = (197.45°, −23.38°) is contained in our 90% and 50% confidence area.
- 90% (50%) area: 50 (11) deg^2



Run time

- SNR time series of 3 detectors, one Intel Xeon CPU E5-2695 core (2.40GHz), Nlevel=6 for adaptive HEALPix sampling, 82-data-point timeseries (20ms 4096Hz data): ~2.4 seconds
- Run time of Bayestar (stated in their paper 1508.03634) : $O(10^2)$ seconds using one Xeon E5 2630 CPU core (2.40GHz)
- Time can be reduced by multi-threads computation

Summary & Acknowledgement

- We proposed a semi-analytical Bayesian method to localize GW sources:
  - Analytical approximation of prior function ->analytical marginalization over matrix **A.**
  - Only need one-fold numerical integral which makes it fast.
- Consistent with PE results, self-consistent.
- The algorithm can be implemented into online CBC detection pipelines, e.g. SPIIR.

Thank you!

We especially thank Takuya Tsutsui, Kipp Cannon, and Leo Singer for their valuable suggestions.